An Accurate Solution of the Cylindrical Dielectric Resonator Problem

MAREK JAWORSKI AND MARIAN W. POSPIESZALSKI

***Abstract*—A cylindrical sample of low-loss high-ϵ_r dielectric placed between two parallel conducting plates perpendicular to the sample axis forms a microwave resonator. A new method of determining the field distribution and resonant frequency of this resonator is presented. By this method the solution is obtained in a form of successive approximations converging to the exact solution. The analysis is outlined in detail for the $TE_{01\delta}$ mode and compared with previously published approximate calculations and experimental data.**

I. Introduction

THE AVAILABILITY of low-loss high-ϵ_r temperature compensated microwave ceramics [1]–[3] allows the construction of dielectric resonators with Q factors and temperature stabilities comparable with those of invar cavities [1],[3]. The use of dielectric resonators in microwave circuits can be expected to expand rapidly since they are compatible with different forms of waveguide structures [3],[4], and they also offer the possibility of integration with different microwave semiconductor devices.

A dielectric resonator structure commonly used in practical microwave circuits is that composed of a cylindrical dielectric sample placed between two parallel conducting plates perpendicular to the sample axis. There have been several approximate calculations of the resonant frequencies of this structure's TE modes [3],[5],[6],[16]. This paper presents an accurate solution of this problem under the following assumptions:

1) the dielectric sample is isotropic and lossless,
2) the plates are perfectly conducting,
3) the distance between the conducting plates is smaller than half of the free-space wavelength corresponding to the resonant frequency of the resonator being calculated.

The third assumption is necessary to assure that the resonant frequencies being calculated are real numbers. Without this assumption, the resonant frequencies could be complex numbers due to radiation, as it is for the dielectric resonator in free space [7],[8].

The analytical approach taken here is accurate in the sense that the solution is obtained in the form of successive approximations converging to the exact solution. Therefore, it allows precise computations of both resonant frequency and field distribution to any desired accuracy.

Manuscript received April 3, 1978; revised October 9, 1978.

M. Jaworski is with the Institute of Physics, Polish Academy of Sciences, Al. Lotnikow 32/46, 02-668 Warszawa, Poland.

M. W. Pospieszalski is with the National Radio Astronomy Observatory, Charlottesville, VA, on leave from the Institute of Electronics Fundamentals, Warsaw Technical University, Nowowiejska 15/19, 00-665 Warszawa, Poland.

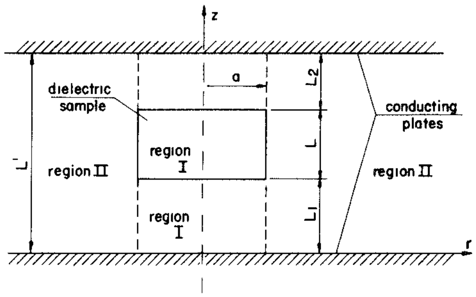

Fig. 1. A dielectric resonator structure.

II. Method of Analysis

The analytical approach is based on the variational method of Weinstein for the approximate calculation of eigenvalues [9]. It recently has been used in the investigation of reentrant cylindrical cavities [10],[11]. For the resonant structure of Fig. 1 the method consists of the following:

1) dividing the space between the metallic plates into two regular regions: I) $r<a$, II) $r>a$,
2) solving Helmholtz equation within each region,
3) matching the resulting fields on the surface $r=a$.

We shall analyze the axially symmetric $TE_{01\delta}$ mode which is the most important mode for practical applications. Although region I ($r<a$) is inhomogeneous along the z axis, the analytical solution will be determined in the form of a Green's function. It may be possible to investigate other modes in a similar manner.

Let $H_z(a,z)$ be the z-component of the magnetic field at the surface $r=a$. According to the equivalence principle [12], one may assume the electric field in each region to be excited by surface currents equivalent to the magnetic field $H_z(a,z)$. Therefore,

$$E_\phi^{\mathrm{I}}(r,z) = -ik_0Z_f\int_0^{L'} G^{\mathrm{I}}(r,a,z,z')H_z(a,z')\,dz' \quad (1a)$$

$$E_\phi^{\mathrm{II}}(r,z) = ik_0Z_f\int_0^{L'} G^{\mathrm{II}}(r,a,z,z')H_z(a,z')\,dz' \quad (1b)$$

where L' is L_1+L+L_2, $G(r,r',z,z')$ is the two-dimensional Green's function for the Helmholtz operator, $H_z(a,z')$ is the z-component of the magnetic field at the surface $r=a$,

0018-9480/79/0700-0639$00.75

$k_0=(2\pi/\lambda_0)$ is the free-space propagation constant, and Z_f is the free-space wave impedance.

In particular, for $r=a$ one obtains

$$E_\phi^{\mathrm{I}}(a,z)=R^{\mathrm{I}}H_z(a,z) \tag{2a}$$

$$E_\phi^{\mathrm{II}}(a,z)=-R^{\mathrm{II}}H_z(a,z) \tag{2b}$$

in which R^{I} and R^{II} denote the integral operators defined by (1). The continuity condition for the electric field tangential to the surface $r=a$ requires that

$$E_\phi^{\mathrm{I}}-E_\phi^{\mathrm{II}}=0 \tag{3a}$$

or

$$R^{\mathrm{I}}H_z(a,z)+R^{\mathrm{II}}H_z(a,z)=0. \tag{3b}$$

The integral equation (3b) is the basic equation from which both the resonant frequency and the field distribution will be found by a procedure similar to the method of moments [13].

Let $\{\psi_j\}$ denote an infinite set of testing functions, complete in the interval $[0,L']$. It may be shown that (3b) is equivalent to the following infinite set of equations:

$$\langle R^{\mathrm{I}}H_z(a,z) \quad \psi_j\rangle+\langle R^{\mathrm{II}}H_z(a,z) \quad \psi_j\rangle=0 \tag{4}$$

where $j=1,2,3,\cdots$ and $\langle\rangle$ denotes the inner product.

The unknown magnetic $H_z(a,z)$ may be expressed in terms of the functions $\{\phi_i\}$:

$$H_z(a,z)=\sum_{i=1}^{\infty} c_i\phi_i(z) \tag{5}$$

where c_i are coefficients to be determined and $\{\phi_i\}$ is a set of functions complete in the interval $[0,L']$. The substitution of (5) into (4) results in the infinite set of homogeneous linear equations of the general form

$$\sum_{i=1}^{\infty} c_i\left[\langle R^{\mathrm{I}}\phi_i,\psi_j\rangle+\langle R^{\mathrm{II}}\phi_i,\psi_j\rangle\right]=0, \quad j=1,2,3,\cdots. \tag{6}$$

A nontrivial solution for c_i exists only if the determinant of the set (6) equals zero. Thus the final equation for the resonant frequency can be written as

$$\det\left[W\right]=0 \tag{7}$$

where the elements of the matrix $[W]$ are given by

$$w_{ij}=\langle R^{\mathrm{I}}\phi_i \quad \psi_j\rangle+\langle R^{\mathrm{II}}\phi_i \quad \psi_j\rangle. \tag{8}$$

Once the resonant frequency is found from (7), the set of linear equations (6) can be solved for c_i to determine the field $H_z(a,z)$ from (5). Consequently, the electric field in the whole region under consideration can be determined from (1).

The exact solution is obtained as the matrix $[W]$ dimension tends to infinity. Neglecting the terms for $i,j>n$ results in an approximate solution. Thus any given accuracy can be obtained by appropriate choice of n. According to the general features of Weinstein's variational method, the consecutive expansion of the matrix $[W]$ dimension n yields a monotonically increasing sequence of approximations. In other words, any approximate solution provides a lower bound for the true resonant frequency. That is,

$$f_0^{(1)}\leqslant f_0^{(2)}\leqslant\cdots\leqslant f_0^{(n)}\cdots<f_0 \tag{9}$$

where f_0 is the exact resonant frequency, and $f_0^{(n)}$ is the approximate solution of (7) for $i,j\leqslant n$.

III. Numerical Results

We shall compute the resonant frequency of the $\mathrm{TE}_{01\delta}$ mode of the resonant structure, shown in Fig. 1, for the case $L_1=L_2$. Because of the symmetry of the problem it is convenient now to move the origin of the coordinate system along the z axis to the point $z=(1/2)L'$, and to analyze the resonant system in the interval $[0,(1/2)L+L_1]$. The function sets $\{\phi_i\}$ and $\{\psi_j\}$ have been chosen to be the same, and have the form

$$\phi_k(z)=\psi_k(z)=\sqrt{\frac{2}{L'}}\cos\left(\frac{\pi}{L'}(2k-1)z\right), \qquad k=1,2,3,\cdots \tag{10}$$

where $L'=L+2L_1$. Each member of the function set matches the boundary conditions and the set is complete for even functions. The Green's functions for regions I and II are given in the Appendix. Thus substituting (10) in (8) one obtains

diagonal elements:

$$\begin{aligned} w_{ii}=&\frac{L}{L'}\frac{J_1(x_i)}{x_iJ_0(x_i)}\left[1+\frac{\sin(\gamma_iL)}{\gamma_iL}\right]\\ &+\left(1-\frac{L}{L'}\right)\frac{I_1(y_i)}{y_iI_0(y_i)}\left[1-\frac{\sin(\gamma_iL)}{\gamma_i(L'-L)}\right]+\frac{K_1(y_i)}{y_iK_0(y_i)}\\ &-\frac{8(\epsilon_r-1)^2k_0^4}{a^2L'}\sum_{m=1}^{\infty}\frac{(P_mA_i+B_i)(Q_mA_i-B_i)}{(u_m^2+\gamma_i^2)(v_m^2+\gamma_i^2)(P_m+Q_m)} \end{aligned} \tag{11a}$$

off-diagonal elements:

$$\begin{aligned} w_{ij}=&\frac{4}{L'}\frac{B_iA_j-B_jA_i}{\gamma_i^2-\gamma_j^2}\left[\frac{J_1(x_i)}{x_iJ_0(x_i)}-\frac{I_1(y_i)}{y_iI_0(y_i)}\right]\\ &-\frac{8(\epsilon_r-1)^2k_0^4}{a^2L'}\sum_{m=1}^{\infty}\frac{1}{(u_m^2+\gamma_i^2)(v_m^2+\gamma_i^2)(P_m+Q_m)}\\ &\cdot\left[\frac{(P_mA_j+B_j)(Q_mA_i-B_i)}{u_m^2+\gamma_j^2}-\frac{(P_mA_i+B_i)(Q_mA_j-B_j)}{v_m^2+\gamma_j^2}\right] \end{aligned} \tag{11b}$$

where, for $k=i,j$:

$$\gamma_k=(2k-1)\frac{\pi}{L'} \qquad A_k=\cos\left(\gamma_k\frac{L}{2}\right)$$

$$B_k=\gamma_k\sin\left(\gamma_k\frac{L}{2}\right) \qquad x_k=a\sqrt{\epsilon_rk_0^2-\gamma_k^2}$$

$$y_k=a\sqrt{\gamma_k^2-k_0^2}$$

TABLE I
THE COMPUTED FREQUENCY OF THE $TE_{01\delta}$ MODE VERSUS DIMENSION n OF THE MATRIX $[W]$ FOR THE RESONATOR HAVING $D=7.99$ mm, $L=2.14$ mm, $L_1/L=2.07$, $\epsilon_r=36.2$

n	1	2	3	4	5
fo [GHz]	7.5519	7.7195	7.7524	7.7578	7.7583
n	6	7	8	9	10
fo [GHz]	7.7583	7.7584	7.7585	7.7585	7.7585

and

$$P_m = u_m \tanh\left(u_m \frac{L}{2}\right) \qquad Q_m = v_m \coth(v_m L_1)$$
$$u_m = \sqrt{h_m^2 - \epsilon_r k_0^2} \qquad v_m = \sqrt{h_m^2 - k_0^2}$$

and h_m is mth root of the equation $J_0(h_m a)=0, \epsilon_r$ is the relative dielectric constant of the sample, J_n is the Bessel function of the first kind, and I_n, K_n are modified Bessel functions of the first and second kind, respectively.

For sufficiently large values of m, the terms of the infinite series in (11a) and (b) decrease as m^{-7} and m^{-5}, respectively. This fast convergence of both series allows the accurate evaluation of the matrix elements.

Expressions (11a) and (b) are valid for $k_0<\gamma_1$, i.e., for distances between conducting plates smaller than the half of the free-space wavelength corresponding to the resonant frequency of the structure. As pointed out earlier, this case represents a closed-resonant system. As a result, the Q factor of the resonator is infinite if the dielectric losses are zero and the plates are perfectly conducting.

To illustrate the theory, the resonant frequencies of several X-band cylindrical dielectric resonators which have been previously investigated experimentally [3] were computed. Since the value of n for a given computational accuracy depends on the geometry, and is not known *a priori*, the computations were performed for successive values of n until the fractional change in the computed resonant frequency was less than 10^{-5}. For the cases considered, this accuracy was obtained for values of n between 8 and 10. An example of the numerical convergence of method is shown in Table I. A comparison between the resonant frequencies of the $TE_{01\delta}$ mode computed by three different approximate methods, this method, and the experimentally obtained data are shown in Table II. The agreement between the accurate solution and the experimental data is within the estimated accuracy of the experimental data [3]. The resonant frequencies computed using a magnetic-wall waveguide model [5] are seen to be about 4–10 percent smaller than the measured values. On the other hand, the dielectric waveguide model [6] and its modification [3] is seen to result in errors up to 7 and 3.5 percent, respectively. The approximate method of Konishi *et al.* has been evaluated

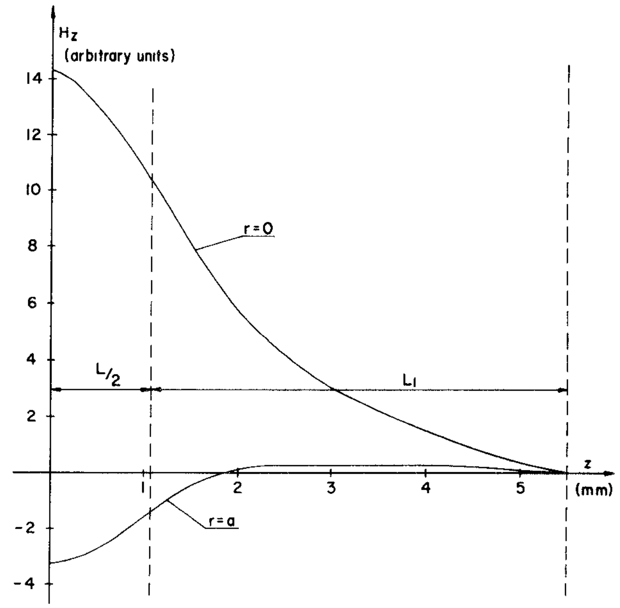

Fig. 2. The distribution of the H_z field component of the $TE_{01\delta}$ mode at the surface $r=a$ and at $r=0$ for the resonator of Table I ($D=2a=7.99$ mm, $L=2$, 14 mm, $L_1/L=2.07$, $\epsilon_r=36.2$).

TABLE II
THE MEASURED AND COMPUTED RESONANT FREQUENCIES ($TE_{01\delta}$ MODE, STRUCTURE AS IN FIG. 1, $L_1=L_2$, $\epsilon_r=36.2$)

D [mm]	L [mm]	L_1/L	fo [GHz] computed ref [5]	ref [6]	ref [3]	this method	measured [3]
4.06	5.15	0.568	10.09	10.86	10.82	10.50	10.48
6.03	4.16	0.820	7.42	8.31	8.20	7.94	7.94
5.98	2.95	1.36	8.03	9.16	8.94	8.61	8.64
6.02	2.14	2.07	8.70	10.08	9.71	9.33	9.40
7.99	2.14	2.07	7.16	8.38	7.96	7.76	7.79

for the dielectric sample in free space [15], for which our method cannot be directly applied. However, it has been shown by Itoh and Rudokas [6] that the method of Konishi *et al.* gives good agreement with their dielectric waveguide model.

The distribution of the H_z field component (for $r=0$ and $r=a$) was computed in accordance with this new method (see (5)) for the resonator of Table I. A graph of this field is shown in Fig. 2. Note that $H_z(a)$ changes its sign along z. Therefore, the magnetic field distribution in the resonator for the $TE_{01\delta}$ mode is similar to the field shown schematically in Fig. 3(c). For comparison, the magnetic field distributions resulting from the magnetic-wall waveguide model Fig. 3(a) and the dielectric waveguide model Fig. 3(b) are also shown.

Table III provides some further insight into the dependence of the computed resonant frequency on different parameters of the structure shown in Fig. 1 (compare [3]).

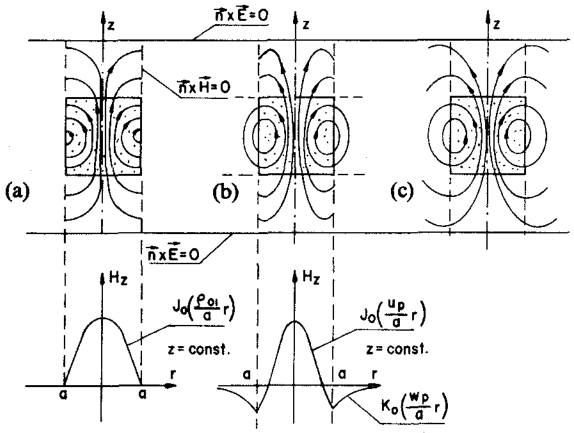

Fig. 3. Comparison between the $TE_{01\delta}$-mode magnetic field distributions resulting from the magnetic wall waveguide model (a) [5] and the dielectric waveguide model (b) [6] with the distribution in the real resonator (c).

IV. Conclusions

The analytical method presented in this paper allows, for the first time, the determination of the resonant frequencies and the field distributions of cylindrical-dielectric resonators with arbitrarily small error. A detailed outline of this method for the $TE_{01\delta}$ mode has been given. This mode is the most important one in practical applications of these resonators. The numerical calculations performed show that not only is the method computationally practical, but also support the claimed 3.5 percent accuracy of the modified dielectric waveguide model [3]. The computed field distribution of the $TE_{01\delta}$ mode shows that the cross-sectional field distribution changes strongly along the z axis. This fact is of importance in the study of the interaction of the dielectric resonator with a ferrite semiconductor device, or other, resonator or waveguide system. Since this method allows calculations with arbitrarily small errors, it can be used in the measurement of complex permittivity of low-loss microwave insulators.

TABLE III
The Computed Values of $(f_0D)^2\epsilon_r$ in [GHz·cm]^2*10^3 for Different Values of D/L, L_1/L, ϵ_r

L_1/L	ϵ_r	$(D/L)^2$ 2.25	4.00	6.25
0.5	30	0.8791	1.0616	1.2690
	100	0.8949	1.0760	1.2829
1.0	30	0.8272	0.9703	1.1234
	100	0.8521	0.9939	1.1470

Appendix

Using Friedman's method [14], the Green's function for the region I is found to be of the following form:

$$G^{I}(r,r',z,z')=\frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_1(h_m r)J_2(h_m r')}{J_1^2(h_m a)}G_m(z,z') \quad \text{(A1)}$$

where G_m is the one-dimensional Green's function given by

$$G_m(z,z')=\frac{g_{1m}(z)g_{2m}(z')}{C_m}, \quad \text{for } z<z' \quad \text{(A2)}$$

$$G_m(z,z')=\frac{g_{1m}(z')g_{2m}(z)}{C_m}, \quad \text{for } z>z' \quad \text{(A3)}$$

and

$$C_m=u_m v_m\left[u_m \sinh\left(\tfrac{1}{2}u_m L\right)\sinh(v_m L_1)+v_m\cosh\left(\tfrac{1}{2}u_m L\right)\cosh(v_m L_1)\right] \quad \text{(A4)}$$

$$g_{1m}(z)=\begin{cases} v_m\cosh(u_m z), & \text{for } 0<z<\tfrac{1}{2}L\\ v_m\cosh\left(\tfrac{1}{2}u_m L\right)\cosh\left[v_m\left(z-\tfrac{1}{2}L\right)\right]+u_m\sinh\left(\tfrac{1}{2}u_m L\right)\sinh\left[v_m\left(z-\tfrac{1}{2}L\right)\right], & \text{for } \tfrac{1}{2}L<z<L_1+\tfrac{1}{2}L\end{cases}$$

$$g_{2m}(z)=\begin{cases} u_m\sinh\left[v_m\left(L_1+\tfrac{1}{2}L-z\right)\right], & \text{for } \tfrac{1}{2}L<z<L_1+\tfrac{1}{2}L\\ u_m\cosh\left[u_m\left(\tfrac{1}{2}L-z\right)\right]\sinh(v_m L_1)+v_m\cosh(v_m L_1)\sinh\left[u_m\left(\tfrac{1}{2}L-z\right)\right], & \text{for } 0<z<\tfrac{1}{2}L.\end{cases} \quad \text{(A5)}$$

All the symbols were defined previously in the text.

The Green's function for region II is derived in a similar way under the condition $k_0<\gamma_i$ (or $L'<\frac{1}{2}\lambda_0$). It means that there is no radiation of electromagnetic energy from this resonant structure. The function G^{II} has the form

$$G^{II}(r,r',z,z')=\frac{2}{L'}\sum_{i=1}^{\infty}\frac{\cos(\gamma_i z)\cos(\gamma_i z')}{K_0(y_i)}\times\begin{cases}\left[I_0(y_i)K_1\left(\frac{y_i}{a}r\right)+K_0(y_i)I_1\left(\frac{y_i}{a}r\right)\right]K_1\left(\frac{y_i}{a}r'\right), & \text{for } r<r'\\ \left[I_0(y_i)K_1\left(\frac{y_i}{a}r'\right)+K_0(y_i)I_1\left(\frac{y_i}{a}r'\right)\right]K_1\left(\frac{y_i}{a}r\right), & \text{for } r>r'.\end{cases} \quad \text{(A6)}$$

All the symbols were defined previously in the text.

Acknowledgment

The authors express their appreciation to Prof. A. K. Smoliński of Warsaw Technical University, Warsaw, Poland, and also to Prof. M. Schetzen of Northeastern University, Boston, MA, for helpful comments.

REFERENCES

[1] K. Wakino *et al.*, "Microwave bandpass filters containing dielectric resonators with improved temperature stability and spurious response," in *Proc. IEEE Int. Microwave Symp.*, Palo Alto, CA, pp. 63–65, May 1975.

[2] D. J. Masse *et al.*, "A new low-loss high-ϵ_r temperature compensated dielectric for microwave applications," *Proc. IEEE*, vol. 59, pp. 1628–1629, Nov. 1971.

[3] M. W. Pospieszalski, "Cylindrical dielectric resonators and their applications in TEM line microwave circuits," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 233–238, Mar. 1979.

[4] A. Karp, H. J. Shaw, and D. K. Winslow, "Circuit properties of microwave dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 818–829, Oct. 1968.

[5] S. Fiedziuszko and A. Jeleński, "The influence of conducting walls on resonant frequencies of the dielectric resonator," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, p. 778, Sept. 1971.

[6] T. Itoh and R. Rudokas, "New method for computing the resonant frequency of dielectric resonator," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 52–54, Jan. 1977.

[7] J. van Bladel, "On the resonances of a dielectric resonator of very high permittivity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 199–208, Feb. 1975.

[8] ——, "The excitation of dielectric resonators of very high permittivity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 208–218, Feb. 1975.

[9] S. H. Gould, *Variational Methods for Eigenvalue Problems*. Toronto, Canada: Univ. of Toronto Press, 1966.

[10] M. Jaworski, "On the resonant frequency of a re-entrant cylindrical cavity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 256–260, Apr. 1978.

[11] ——, "Resonant cavity Method for the Determination of Complex Permittivity", (in Polish), Ph.D. dissertation, Inst. of Phys., Polish Academy of Sciences, Warsaw, Poland, 1976.

[12] R. F. Harrington, *Time Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.

[13] ——, *Field Computation by Moment Methods*. New York: MacMillan, 1968.

[14] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Wiley, 1956.

[15] Y. Konishi, N. Hoshino, and Y. Utsumi, "Resonant frequency of a $TE_{01\delta}$ dielectric resonator," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 112–114, Feb. 1976.

[16] P. Guillon and Y. Garault, "Accurate resonant frequencies of dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 916–922, Nov. 1977.

Design of Microwave GaAs MESFET's for Broad-Band Low-Noise Amplifiers

HATSUAKI FUKUI, SENIOR MEMBER, IEEE

***Abstract*—As a basis for designing GaAs MESFET's for broad-band low-noise amplifiers, the fundamental relationships between basic device parameters, and two-port noise parameters are investigated in a semiempirical manner. A set of four noise parameters are shown as simple functions of equivalent circuit elements of a GaAs MESFET. Each element is then expressed in a simple analytical form with the geometrical and material parameters of this device. Thus practical expressions for the four noise parameters are developed in terms of the geometrical and material parameters.**

Among the four noise parameters, the minimum noise figure $F_{\min}$, and equivalent noise resistance R_n, are considered crucial for broad-band low-noise amplifiers. A low R_n corresponds to less sensitivity to input mismatch, and can be obtained with a short heavily doped thin active channel. Such a high channel doping-to-thickness (N/a) ratio has a potential of producing high power gain, but is contradictory to obtaining a low $F_{\min}$. Therefore, a compromise in choosing N and a is necessary for best overall amplifier performance. Four numerical examples are given to show optimization processes.

I. INTRODUCTION

THE GaAs Schottky-barrier gate field effect transistors (GaAs MESFET's) have demonstrated excellent noise and gain performance at microwave frequencies through K band [1]. The excellent microwave performance of GaAs MESFET's is certainly related to their channel properties. GaAs MESFET's to be used for broad-band low-noise amplifier applications, must have special requirements on their channel properties for optimum performance. The purpose of this paper is to investigate the fundamental relationships between the noise and small-signal properties, and the basic channel parameters of GaAs MESFET's. This information should be useful as a basis for device design.

Manuscript received August 14, 1978; revised January 15, 1979.

The author is with Bell Laboratories, Murray Hill, NJ 07974.

II. REPRESENTATION OF NOISE PROPERTIES

A. Noise Parameters

From the circuit point of view, the GaAs MESFET can be treated as a black box of noisy two port. The noise properties of such a black box are then characterized by a set of four noise parameters in the binomial form [2]. A derivation of this form can be written as

$$F = F_{\min} + \frac{R_n}{R_{ss}}\left[\frac{(R_{ss}-R_{op})^2+(X_{ss}-X_{op})^2}{R_{op}^2+X_{op}^2}\right] \quad (1)$$

0018-9480/79/0700-0643$00.75